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Nonlinear Electrohydrodynamic Rayleigh–Taylor Instability with Mass and Heat Transfer Subject to a Vertical Oscillating Force and a Horizontal Electric Field

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Weakly nonlinear stability of interfacial waves propagating between two electrified inviscid fluids influenced by a vertical periodic forcing and a constant horizontal electric field is studied. Based on the method of multiple-scale expansion for a small-amplitude periodic force, two parametric nonlinear Schrödinger equations with complex coefficients are derived in the resonance cases. A standard nonlinear Schrödinger equation with complex coefficients is derived in the nonresonance case. A temporal solution is carried out for the parametric nonlinear Schrödinger equation. The stability analysis is discussed both analytically and numerically.

1. INTRODUCTION

Because of the wide range of important industrial applications for the Rayleigh–Taylor instability, there has been a growing interest in recent years in the study of interfacial waves of a fluid subject to a horizontal or vertical oscillation from the viewpoint of an important variety of scientific and technical problems. Very few theoretical studies have been done to understand the stage of interfacial waves when subject to external periodic force. Based on a linear theory, Benjamin and Ursell (1954) explained the excitation of standing waves of an inviscid liquid associated with the instability of the Mathieu equation for a parametric resonant mode.

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Kelly (1965) considered the effect of an oscillatory component in the basic velocity on the stability of the classical Kelvin–Helmholtz profile. He deduced that when the differences in the mean speed are below the steady critical speed for instability but are long compared with the amplitude of the fluctuations, parametric amplification of waves at the interface occurs.

Wu *et al.* (1984) studied experimentally the subharmonic excitation of solitary waves in a long channel subject to a vertical periodic oscillation with an appropriate frequency and amplitude. Ciliberto and Gollub (1985) carried out an experiment on parametric excitation of a cylindrical fluid layer in a circular vessel. They studied excitation of a pair of noncircularly symmetric modes and found an interaction between the two modes in which the wave pattern oscillates either periodically or chaotically with a period long compared with that of the forcing.

Roberts (1973) considered the stability of an unsteady basic flow of a conducting fluid in the presence of a parallel magnetic field. The particular profile investigated is the classical Kelvin–Helmholtz profile modified by the addition of an oscillatory component. Two cases are considered in detail: that of a perfectly conducting fluid and that of a poorly conducting fluid. The investigation leads in both cases to an equation of the Hill type. It is concluded that the magnetic field has a stabilizing influence, but is nevertheless unable to suppress the Kelvin–Helmholtz instability in an unsteady (basic) flow.

Umeki and Kambe (1989; Kambe and Umeki, 1990) studied surface waves in a closed container subject to a vertical oscillation. The external forcing is equivalent to oscillation of the acceleration of gravity.

Very few studies on nonlinear electrohydrodynamic Rayleigh–Taylor instability have been attempted. Mohamed and El Shehawey (1983a, b, 1984) studied the nonlinear instability of an interface between two fluids under the influence of a periodic electric field.

El-Dib (1993) studied nonlinear wave propagation on the surface between two superposed magnetic fluids stressed by a tangential periodic magnetic field. A stability analysis reveals the existence of both nonresonant and resonant cases. He found that the tangential periodic magnetic field plays a dual role in the stability criterion, while the field frequency has a destabilizing influence.

More recently current interest in microgravity material processing has focused attention upon certain relevant aspects of fluid mechanics in this environment. In particular a number of materials processing applications involve a fluid–fluid interface.

Lyell and Roh (1991) studied the effect of a periodic acceleration on the interface stability of an idealized fluid configuration.

The fluid configuration is multilayered and infinite in extent. El-Dib (1994) considered a theoretical analysis of the subharmonic response of two resonant modes of the interfacial gravity-capillary waves between two electrified fluids of infinite depth under the influence of a constant horizontal electric field. The method of multiple scales was used to derive two parametrically nonlinear Schrödinger equations which describe the behavior of the disturbed system in the resonance case. One of them contains the first derivatives in space for a complex conjugate type, while the second contains a linear complex conjugate term. A time-dependent solution of a traveling wave was obtained. He found that the stability criteria are significantly affected by the amplitude of the temporal solution. The acceleration frequency plays a dual role in the stability criterion. The results showed that the horizontal electric field plays a dual role in the resonance case.

The aim of the work presented here is to extend the approaches of El-Dib (1994) in order to examine theoretically, through a nonlinear perturbation analysis, the effect of periodic acceleration to a horizontal interface admitting mass and heat transfer. We have therefore considered the Rayleigh–Taylor instability problem with mass and heat transfer in a plane geometry using Hsieh's (1979) simplified formulation. This formulation has been successfully used by Mohamed *et al.* (1993). They studied nonlinear electrohydrodynamic stability of two superposed dielectric fluids with interfacial mass and heat transfer for layers of finite thickess. The fluids are subjected to a constant tangential electric field. The stability criterion is expressible in terms of various competing parameters representing the equilibrium heat flux, latent heat evaporation, gravity, surface tension, densities of the fluids, dielectric constants of the fluids, thickness of the layers, and thermal properties of the fluids.

2. BASIC EQUATIONS

Consider two dielectric, inviscid, incompressible fluids confined between two parallel planes $y = -h_1$ and $y = h_2$. The interface is given by

$$S(x, y, t) = y - \zeta(x, t) = 0 \tag{1}$$

where $y = 0$ represents the equilibrium interface. The fluid of density $\rho^{(1)}$, dielectric constant $\tilde{\epsilon}^{(1)}$, and depth h_1 occupies the region $y < 0$, whereas the medium $y > 0$ is occupied by the fluid of density $\rho^{(2)}$, dielectric constant $\tilde{\epsilon}^{(2)}$, and depth h_2. The temperatures at $y = h_2$, $y = -h_1$, and $y = 0$ are $T^{(2)}$, $T^{(1)}$, and $T^{(0)}$, respectively. The fluids are subjected to an external electric field E_0 acting in the x direction. We work in rectangular coordinates (x, y)

with the x axis aligned with the mean interface level. The system here is stressed by a periodic acceleration in the negative y direction, i.e.,

$$\mathbf{G} = -(g + \epsilon g_0 \cos \omega_0 t)\mathbf{e}_y \tag{2}$$

The flow is assumed to be irrotational and two-dimensional in each layer. The basic equations governing the perturbed velocity potential $\phi(\mathbf{v} = \nabla\phi)$ are

$$\nabla^2\phi^{(1)} = 0 \qquad \text{for} \quad -h_1 < y < \zeta(x, t) \tag{3}$$

$$\nabla^2\phi^{(2)} = 0 \qquad \text{for} \quad \zeta(x, t) < y < h_2 \tag{4}$$

where $y = \zeta(x, t)$ is the elevation of the free surface, and $\nabla \equiv (\partial/\partial x, \partial/\partial y, 0)$.

The perturbation produces an additional electric field, which we assume to be derived from a potential $\psi(x, y)$ satisfying the equations

$$\nabla^2\psi^{(1)} = 0 \qquad \text{for} \quad -h_1 < y < \zeta(x, t) \tag{5}$$

$$\nabla^2\psi^{(2)} = 0 \qquad \text{for} \quad \zeta(x, t) < y < h_2 \tag{6}$$

$$\mathbf{E} = \mathbf{E}_0 - \nabla\psi \tag{7}$$

Here ϕ and ψ are, respectively, the velocity potential and electrostatic potential perturbations. The velocity and electrostatic potentials satisfy the conditions

$$\left[\frac{\partial\phi^{(1)}}{\partial y}\right]_{y=-h_1} = \left[\frac{\partial\phi^{(2)}}{\partial y}\right]_{y=h_2} = 0 \tag{8}$$

$$\left[\frac{\partial\psi^{(1)}}{\partial y}\right]_{y=-h_1} = \left[\frac{\partial\psi^{(2)}}{\partial y}\right]_{y=h_2} = 0 \tag{9}$$

The interfacial boundary conditions between the two fluids are as follows:

1. The tangential component of the electric field is continuous at the interface,

$$\left[\left[\frac{\partial\psi}{\partial x}\right]\right] + \frac{\partial\zeta}{\partial x}\left[\left[\frac{\partial\psi}{\partial y}\right]\right] = 0 \qquad \text{at} \quad y = \zeta \tag{10}$$

2. The normal electric displacement is continuous at the interface

$$E_0 \frac{\partial\zeta}{\partial x}[[\tilde{\epsilon}]] - \frac{\partial\zeta}{\partial x}\left[\left[\tilde{\epsilon}\frac{\partial\psi}{\partial x}\right]\right] + \left[\left[\tilde{\epsilon}\frac{\partial\psi}{\partial y}\right]\right] = 0 \qquad \text{at} \quad y = \zeta \tag{11}$$

where $[[\cdot]]$ represents the jump across the interface.

We can express the stress tensor as

$$\Pi_{ij} = -\Pi\delta_{ij} + \tilde{\epsilon}E_iE_j - (1/2)\tilde{\epsilon}E^2\delta_{ij} \tag{12}$$

where $\Pi = p - (1/2)\bar{\epsilon}E_0^2$, where p is the hydrostatic pressure, which can be obtained from Bernoulli's equation, and δ_{ij} is the Kronecker delta. Then the interfacial conditions which express the conservation of mass and momentum are given by

$$\rho^{(1)}\left(\frac{\partial S}{\partial t} + \nabla\phi^{(1)} \cdot \nabla S\right) = \rho^{(2)}\left(\frac{\partial S}{\partial t} + \nabla\phi^{(2)} \cdot \nabla S\right) \quad \text{at} \quad y = \zeta \quad (13)$$

and

$$\rho^{(1)}\left(\frac{\partial S}{\partial t} + \nabla\phi^{(1)} \cdot \nabla S\right)(\nabla\phi^{(1)} \cdot \nabla S)$$
$$= \rho^{(2)}\left(\frac{\partial S}{\partial t} + \nabla\phi^{(2)} \cdot \nabla S\right)(\nabla\phi^{(2)} \cdot \nabla S)\Bigg[(\Pi_{ij}^{(1)} - \Pi_{ij}^{(2)})n_i$$
$$-n_j\sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\Bigg], \quad j = 1, 2, \quad \text{at} \quad y = \zeta \quad (14)$$

where $\mathbf{n}$ is the unit normal vector to the interface, σ is the surface tension coefficient, and R_1 and R_2 are the two principal radii of curvature of the interface. The radius of curvature is taken to be positive if the center of curvature lies on the side of fluid 2, and negative otherwise.

Finally, the interfacial condition for energy transfer is given by

$$L\rho^{(1)}\left(\frac{\partial S}{\partial t} + \nabla\phi^{(1)} \cdot \nabla S\right) = f(\zeta) \quad \text{at} \quad y = \zeta \quad (15)$$

The left-hand side of (15) represents the net heat flux from the interface into the fluid regions when such a phase transformation is taking place. This quantity is taken to be approximately expressible in terms of the balance of heat fluxes in the fluid regions as if the system were instantaneously in dynamic equilibrium. Let us express $f(\zeta)$ in terms of a power series expansion of ζ. To the third order of ζ, we can write

$$f(\zeta) = \alpha(\zeta + \alpha_2\zeta^2 + \alpha_3\zeta^3) \quad (16)$$

Note that according to the quasiequilibrium approximation (Hsieh, 1979), the coefficients α, α_2, and α_3 are given by

$$\alpha = \frac{G}{L}\left(\frac{1}{h_1} + \frac{1}{h_2}\right)$$
$$\alpha_2 = \left(\frac{1}{h_2} - \frac{1}{h_1}\right)$$
$$\alpha_g = \frac{h_1^3 + h_2^3}{h_1^2h_2^2(h_1 + h_2)}$$

where L is the latent heat released when the fluid is transformed from phase 1 to phase 2 and

$$G = \frac{K^{(2)}(T^{(0)} - T^{(2)})}{h_2} = \frac{K^{(1)}(T^{(1)} - T^{(0)})}{h_1}$$

is the equilibrium heat flux.

Here $K^{(1)}$ and $K^{(2)}$ represent the lower and upper thermal conductivities, respectively.

If fluid 1 is hotter than fluid 2, then L is positive and G is positive since $T^{(1)} > T^{(0)} > T^{(2)}$. If fluid 2 is hotter than fluid 1, then L and G are both negative. Therefore, in both cases α is always positive.

Equations (3)–(6) and conditions (10)–(15) constitute the governing equations of the problem.

To investigate the nonlinear interaction of small- but finite-amplitude waves, we apply the method of multiple scales. To that end, we expand the various variables in ascending powers in terms of a small dimensionless parameter ϵ characterizing the amplitude of the periodic force. The independent variables x, t are scaled in a like manner,

$$T_n = \epsilon^n t, \qquad X_n = \epsilon^n x, \qquad n = 0, 1, 2 \tag{17}$$

$$\zeta(x, t) = \sum_{n=1}^{3} \epsilon^n \zeta_n(X_0, X_1, X_2, T_0, T_1, T_2) + O(\epsilon^4) \tag{18}$$

and the variables may expanded as

$$\Phi(x, t) = \sum_{n=1}^{3} \epsilon^n \Phi_n(X_0, X_1, X_2, T_0, T_1, T_2) + O(\epsilon^4) \tag{19}$$

where Φ can be any of the physical quantities ϕ, ψ.

Since the boundary conditions (10), (11), and (14)–(15) are prescribed at the interface $y = \zeta\ (x, t)$, we express all the physical quantities involved in terms of Maclaurin series about $y = 0$. On putting expansions (17) and (18) into the set of equations (3)–(15) and equating the coefficients of equal powers in ϵ, we obtain the linear as well as the successive higher order equations. The hierarchy of equations for each order can be obtained with the knowledge of the previous orders. The solution of the first-order problem leads to the dispersion relation

$$\begin{aligned} F(\omega, k) = {} & (\omega^2/k)(\rho^{(2)} \coth kh_2 + \rho^{(1)} \coth kh_1) + (i\alpha\omega/k) \\ & \times (\coth kh_2 + \coth kh_1) - [E_0^2/\epsilon^*(k)]k(\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)})^2 \\ & \times \cosh kh_2 \cosh kh_1 - g(\rho^{(1)} - \rho^{(2)}) - k^2\sigma = 0 \end{aligned} \tag{20}$$

where

$$\epsilon^*(k) = \tilde{\epsilon}^{(1)} \sinh kh_1 \cosh kh_2 + \tilde{\epsilon}^{(2)} \sinh kh_2 \cosh kh_1$$

The dispersion relation (20) is reduced to

$$a_2(-i\omega)^2 + a_1(-i\omega) + a_0 = 0 \tag{21}$$

where

$$\begin{aligned} a_2 &= \rho^{(1)} \coth kh_1 + \rho^{(2)} \coth kh_2 \\ a_1 &= \alpha(\coth kh_1 + \coth kh_2) \\ a_0 &= \{E_0^2 k^2 (\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)})^2 / (\tilde{\epsilon}^{(1)} \tanh kh_1 + \tilde{\epsilon}^{(2)} \tanh kh_2)\} \\ &\quad + k^3\sigma + gk(\rho^{(1)} - \rho^{(2)}) \end{aligned}$$

We know from the Routh–Hurwitz criterion (Zahreddine and El Shehawey, 1988) that necessary and sufficient conditions for stability for the quadratic equation (21) are

$$a_1 > 0 \qquad \text{and} \qquad a_0 > 0 \tag{22}$$

since a_2 is always positive.

The conditions $a_1 > 0$ is trivially satisfied since $\alpha > 0$, while the condition $a_0 > 0$ gives

$$\begin{aligned} E_0^2 > &\{[\tilde{\epsilon}^{(1)} \tanh kh_1 + \tilde{\epsilon}^{(2)} \tanh kh_2][g(\rho^{(2)} - \rho^{(1)}) - k^2\sigma]\} \\ &\times \{k(\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)})^2\}^{-1} \end{aligned} \tag{23}$$

It is clear from (23) that the tangential electric field is stabilizing. For values of $E_0 > E_C$, where

$$\begin{aligned} E_C^2 = &\{[\tilde{\epsilon}^{(1)} \tanh kh_1 + \tilde{\epsilon}^{(2)} \tanh kh_2][g(\rho^{(2)} - \rho^{(1)}) - k^2\sigma]\} \\ &\times \{k(\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)})^2\}^{-1} \end{aligned} \tag{24}$$

the system is linearly stable. The periodic acceleration has no effect in this order.

3. THE SECOND-ORDER PROBLEM

The inclusion of a periodic force across the interface yields results radically different from the classical case. In the classical Rayleigh–Taylor problem, upon admitting mass and heat transfer, the second-order surface deflection ζ_2 is modified to be

$$\zeta_2 = \zeta_{20} + \zeta_{11} \tag{25}$$

where ζ_{20} represents the elevation in the absence of the periodic acceleration as obtained by Mohamed *et al.* (1993),

$$\zeta_{20} = -2\alpha_2 A\bar{A} + \Omega A^2(X_1, X_2, T_1, T_2)\, e^{2i(kX_0 - \omega T_0)} + \text{C.C.} \tag{26}$$

The additional term ζ_{11} is due to the periodic force and is given by

$$\zeta_{11} = \Omega_1(X_1, X_2, T_1, T_2)\, e^{i(kX_0 - \omega T_0)} + \text{C.C.} \tag{27}$$

Here ζ_{11} denotes the forcing due to oscillation of gravity, which is proportional to g_0 and the frequency ω_0. It is found that Ω_1 should satisfy

$$\begin{aligned}
&[(\rho^{(1)} \coth kh_1 + \rho^{(2)} \coth kh_2) \frac{\partial^2}{\partial T_0^2} \\
&+ \alpha (\coth kh_1 + \coth kh_2) \frac{\partial}{\partial T_0} \\
&+ (\rho^{(1)} \coth kh_1 + \rho^{(2)} \coth kh_2)(\omega_r^2 - \omega_i^2) \\
&- \alpha(\coth kh_1 + \coth kh_2)\omega_i]\Omega_1 \\
&= \Bigg[-g_0 k(\rho^{(1)} - \rho^{(2)})A \cos \omega_0 T_0 + 2i\omega_r(\rho^{(1)} \coth kh_1 + \rho^{(2)} \coth kh_2) \\
&\times \left(\frac{\partial A}{\partial T_1} + \frac{d\omega}{dk} \frac{\partial A}{\partial X_1} \right) \Bigg] e^{i(kX_0 - \omega T_0)} + \text{C.C.}
\end{aligned} \tag{28}$$

Equation (28) contains terms which correspond to the factor exp $(-i\omega_r T_0)$. The elimination of these terms leads to secular terms that lead to the solvability conditions. In omitting these terms, we need to distinguish between two cases, the case when the external frequency ω_0 is away from the real part of the wave frequency ω (the nonresonant case) and the case that arises when the frequency ω_0 approaches $2\omega_r$ ($\omega = \omega_r + i\omega_i$). Thus in the nonresonance case the following solvability condition is obtained:

$$\frac{\partial A}{\partial T_1} + \frac{d\omega}{dk} \frac{\partial A}{\partial X_1} = 0 \tag{29}$$

where $(d\omega/dk)$ is the group velocity.

With the solvability condition (29) in the nonresonant case the particular solution of (28) is

$$\begin{aligned}
\Omega_1 = {} & [g_0 kA(\rho^{(1)} - \rho^{(2)})/2\omega_0 L_1(\omega_0^2 - 4\omega_r^2)][\omega_0 \cos \omega_0 T_0 \\
& - 2i\omega_r \sin \omega_0 T_0] A e^{-i\omega T_0} + \text{C.C.}
\end{aligned} \tag{30}$$

In the resonance case the frequency ω_0 is assumed to approach $2\omega_r$; we introduce the detuning parameter σ_1

$$\omega_0 = 2\omega_r + 2\epsilon\sigma_1 \tag{31}$$

and hence

$$-i(\omega_0 - \omega_r)T_0 = -i(\omega_r + 2\epsilon\sigma_1)T_0 \tag{32}$$

Thus the solvability condition in this case yields

$$\frac{\partial A}{\partial T_1} + \frac{d\omega}{dk}\frac{\partial A}{\partial X_1} = i\gamma_0\bar{A}e^{-2i\sigma_1 T_1} \tag{33}$$

where

$$\gamma_0 = [g_0 k(\rho^{(2)} - \rho^{(1)})/4\omega_r L_1] \tag{34}$$

where L_1 is given in the Appendix.

Equation (33) is the solvability condition in the resonant case. Therefore the particular solution of equation (28) is

$$\Omega_1 = \{[g_0 kA(\rho^{(1)} - \rho^{(2)})]/2\omega_0[(\omega_0 + 2\omega)L_1 + iL_2]\}e^{-i(\omega_0+\omega)T_0} + \text{C.C.} \tag{35}$$

where L_2 is given in the Appendix.

4. THE THIRD-ORDER PROBLEM

In this order of the investigation the surface deflection satisfies the following equation:

$$\Bigg[(\rho^{(1)} - \rho^{(2)})g + \frac{L_1}{k}\frac{\partial^2}{\partial T_0^2} + \frac{L_2}{k}\frac{\partial}{\partial T_0} + \sigma k^2 + \frac{kE_0^2(\bar{\epsilon}^{(2)} - \bar{\epsilon}^{(1)})^2}{\epsilon^*(k)}$$
$$\times \cosh kh_1 \cosh kh_2\Bigg]\zeta_3$$
$$= -\Bigg(\frac{i}{k^2}[L_1 + k(\rho^{(2)}h_2 \coth^2 kh_2$$
$$+ \rho^{(1)}h_1 \coth^2 kh_1) - k(\rho^{(2)}h_2 + \rho^{(1)}h_1)]\frac{\partial^3}{\partial T_0^2 \partial X_1} + \Bigg[\frac{i}{k^2}L_2 + \frac{i\alpha}{k}$$
$$\times (h_2 \coth^2 kh_2 + h_1 \coth^2 kh_1) + \frac{2L_1}{k} - \frac{i\alpha}{k}(h_2 + h_1)\Bigg]\frac{\partial^2}{\partial T_0 \partial X_1}$$

$$
\begin{aligned}
&+ \frac{2L_1}{k}\frac{\partial^2}{\partial T_0 \partial T_1} + \frac{L_2}{k}\frac{\partial}{\partial \mathrm{T}_1} + \mathrm{i}\{-2k\sigma - \frac{k(\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)})^2}{\epsilon^{*2}(k)E_0^2} \\
&\times [\epsilon^*(\mathrm{k}) \cosh kh_1 \cosh kh_2 + (\tilde{\epsilon}^{(1)}h_1 \cosh^2 kh_2 + \tilde{\epsilon}^{(2)}h_2 \cosh^2 kh_1)]\} \\
&\times \frac{\partial}{\partial X_1}\,\Omega_1 e^{ikx_0} - \left(-\frac{i}{k}(iL_2 + 2\omega L_1)\frac{\partial A}{\partial T_2} + i\left\{-\frac{iL_2\omega}{k^2} - \frac{\omega^2 L_1}{k^2}\right.\right. \\
&- 2\sigma k + \frac{k(\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)})^2}{\epsilon^{*3}(k)} E_0^2[(\tilde{\epsilon}^{(1)}h_1 \cosh^2 kh_2 + \tilde{\epsilon}^{(2)}h_2 \cosh^2 kh_1) \\
&- \cosh kh_1 \cosh kh_2] - \frac{h_2}{k \sinh^2 kh_2}(i\alpha\omega + \omega^2\rho^{(2)}) - \frac{h_1}{k \sinh^2 kh_1} \\
&\left.\times (i\alpha\omega + \omega^2\rho^{(1)})\right\}\frac{\partial A}{\partial X_2} + \frac{L_1}{k}\frac{\partial^2 A}{\partial T_1^2} + \left[\frac{iL_2}{k^2} + \frac{2\omega L_1}{k^2}\right. \\
&\left.+ \frac{h_1}{k \sinh^2 kh_1}(i\alpha + 2\omega\rho^{(1)}) + \frac{h_2}{k \sinh^2 kh_2}(i\alpha + 2\omega\rho^{(2)})\right)\frac{\partial^2 A}{\partial X_1 \partial T_1} \\
&+ \frac{i\omega L_2}{k^3} + \frac{\omega^2 L_1}{k^3} - \sigma + \left(\frac{h_1}{k^2 \sinh^2 kh_1} + \frac{h_1^2 \cosh kh_1}{k \sinh^3 kh_1}\right) \\
&\times (i\alpha\omega + \omega^2\rho^{(1)}) + \left(\frac{h_2}{k^2 \sinh^2 kh_2} + \frac{h_2^2 \cosh kh_2}{k \sinh^3 kh_1}\right) \\
&\times (i\alpha\omega + \omega^2\rho^{(2)}) + \frac{E_0^2(\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)})^2}{\epsilon^{*3}(k)}[-2kh_1h_2\tilde{\epsilon}^{(2)} \\
&\times \tilde{\epsilon}^{(1)} \cosh kh_1 \cosh kh_2 \\
&- kh_1^2 \cosh^2 kh_2(\tilde{\epsilon}^{(1)} \cosh kh_1 \cosh kh_2 \\
&+ \tilde{\epsilon}^{(2)} \sinh kh_1 \sinh kh_2) \\
&- kh_2^2 \cosh^2 kh_1(\tilde{\epsilon}^{(1)} \sinh kh_1 \sinh kh_2 \\
&+ \tilde{\epsilon}^{(2)} \cosh kh_1 \cosh kh_2) + \epsilon^*(k)(\tilde{\epsilon}^{(1)}h_1 \cosh^2 kh_2 + \tilde{\epsilon}^{(2)} \\
&\left.\times h_2 \cosh^2 kh_1)]\frac{\partial^2 A}{\partial X_1^2} - A^2\bar{A}\Theta\right) e^{i(kX_0 - \omega T_0)} + \mathrm{C.C.} + \mathrm{NST}
\end{aligned}
$$

where Θ is given in the Appendix and NST stands for terms that do not produce secular terms. To analyze the particular solution for equation (36), we need to avoid the nonuniformity in it. Thus, we need the secular terms to vanish. Two possible cases that produce secular terms are (1) ω_0 is away from ω_r and (2) ω_0 approaches $2\omega_r$. The elimination of the secular terms for each case produces its corresponding solvability condition.

4.1. Nonresonance Case

In this case the frequency ω_0 is considered to be away from the wave frequency ω_r. Thus the solvability condition of the third order in the nonresonance case is given by

$$i\frac{\partial A}{\partial T_2} + i\frac{d\omega}{dk}\frac{\partial A}{\partial X_2} + \frac{1}{2}\frac{d^2\omega}{dk^2}\frac{\partial^2 A}{\partial X_1^2} + M_1 A - (Q_1 + iQ_2)A^2\overline{A} = 0 \quad (37)$$

where

$$M_1 = \frac{(\rho^{(1)} - \rho^{(2)})^2 g_0^2}{4\omega_r(\rho^{(1)}\coth kh_1 + \rho^{(2)}\coth kh_2)(\omega_0^2 + 4\omega_r^2)} \quad (38)$$

$$Q_1 = -[k\Theta_r/2(\rho^{(1)}\coth kh_1 + \rho^{(2)}\coth kh_2)\omega_r] \quad (39)$$

$$Q_2 = -[k\Theta_i/2(\rho^{(1)}\coth kh_1 + \rho^{(2)}\coth kh_2)\omega_r] \quad (40)$$

By using the Gardner–Morikawa transformation, we find that equation (37) becomes

$$i\frac{\partial A}{\partial \tau} + (P_1 + iP_2)\frac{\partial^2 A}{\partial \zeta^2} + M_1 A = (Q_1 + iQ_2)A^2\overline{A} \quad (41)$$

where

$$P_1 + iP_2 = \frac{1}{2}\frac{d^2\omega}{dk^2} \quad (42)$$

The solutions of equation (41) are stable if

$$P_1Q_1 + P_2Q_2 > 0 \quad \text{and} \quad Q_2 < 0 \quad (43)$$

The stability in this version of the problem is discussed in Mohamed *et al.* (1993).

4.2. Resonance Case

Inspection of the right-hand side of equation (36) reveals that in addition to terms proportional to the factor $\exp(\pm i\omega_r)\, T_0$, secular terms are produced by the terms proportional to the factor $\exp[\pm i(\omega_0 - \omega_r)\, T_0]$. In this case we express the nearness of ω_0 to $2\omega_r$ by introducing a detuning parameter σ_1 defined according to equation (31); therefore $\exp[-i(\omega_0 - \omega_r)\, T_0] = \exp$

$i(-\omega_r T_0 - 2i\sigma_1 T_1)$. The secular terms are eliminated from equation (36) with the help of equation (33). Thus the solvability condition in this case is given by

$$i\frac{\partial A}{\partial T_2} + i\frac{d\omega}{dk}\frac{\partial A}{\partial X_2} + \frac{1}{2}\frac{d^2\omega}{dk^2}\frac{\partial^2 A}{\partial X_1^2} - (Q_1 + iQ_2)A^2\overline{A} + RA$$

$$+ \left[(S_1 + iS_2)\frac{\partial \overline{A}}{\partial X_1} + F_1\overline{A}\right]e^{-2i\sigma_1 T_1} = 0 \tag{44}$$

where R, S_1, S_2, F_1, M_2, and M_3 are given in the Appendix. By using the Gardner–Morikawa transformation, we find that equation (44) becomes:

$$i\frac{\partial A}{\partial \tau} + (P_1 + iP_2)\frac{\partial^2 A}{\partial \zeta^2} - (Q_1 + iQ_2)A^2\overline{A} + RA + \left[(S_1 + iS_2)\frac{\partial \overline{A}}{\partial \zeta}\right.$$

$$\left. + F_1\overline{A}\right]e^{-2i\sigma_1\epsilon^{-1}\tau} = 0 \tag{45}$$

5. STABILITY ANALYSIS

The above form of a parametric nonlinear Schrödinger equation is new. The stability criterion has not been obtained before. The absence of the parts P_2, Q_2, and S_1 in equation (45) reduces the equation to the same type as those obtained by El-Dib (1993, 1994). We follow the procedure adopted there. Thus we assume that equation (45) admits the following time-dependent solution:

$$A = me^{-i(\sigma_1\epsilon^{-1} - R_1 - Q_1 m^2)\tau} \tag{46}$$

Substituting from equation (46) into equation (45), we get

$$(\sigma_1/\epsilon) + im^2 Q_2 + F_1 \exp[-2i(R + Q_1 m^2)]\tau = 0 \tag{47}$$

By separating real and imaginary parts of equation (47), we obtain

$$(\sigma_1/\epsilon) + F_1\cos[2(R + Q_1 m^2)]\tau = 0 \tag{48}$$

$$m^2 Q_2 - F_1\sin[2(R + Q_1 m^2)]\tau = 0 \tag{49}$$

Squaring equations (48) and (49) and adding, we obtain

$$m^4 = [F_1^2 - (\sigma_1^2/\epsilon^2)]/Q_2^2 \tag{50}$$

m^2 is real when

$$F_1^2 - (\sigma_1^2/\epsilon^2) > 0$$

or

$$m^2 = [F_1^2 - (\sigma_1^2/\epsilon^2)]^{1/2}/Q_2 \tag{51}$$

The solution (46) must be bounded; this requires that

$$Q_2[F_1^2 - (\sigma_1^2/\epsilon^2)]^{1/2} > 0 \tag{52}$$

To investigate the stability of the waves, we perturb the solution (46) according to

$$A = [m + \alpha(\xi, \tau) + i\beta(\xi, \tau)]e^{-i\sigma_1\epsilon^{-1}\tau} \tag{53}$$

where α and β are real. Substituting (53) into (45) and neglecting nonlinear terms in α and β, we get

$$-\frac{\partial\beta}{\partial\tau} + P_1\frac{\partial^2\alpha}{\partial\xi^2} - P_2\frac{\partial^2\beta}{\partial\xi^2} + 2Q_1m^2\alpha - \frac{1}{F_1}\left[\frac{\sigma_1}{\epsilon}\left(S_1\frac{\partial\alpha}{\partial\xi} + S_2\frac{\partial\beta}{\partial\xi}\right)\right.$$
$$\left. - m^2Q_2\left(-S_1\frac{\partial\beta}{\partial\xi} + S_2\frac{\partial\alpha}{\partial\xi} - 2\beta F_1\right)\right] = 0 \tag{54}$$

and

$$\frac{\partial\alpha}{\partial\tau} + P_1\frac{\partial^2\beta}{\partial\xi^2} + P_2\frac{\partial^2\alpha}{\partial\xi^2} + 2Q_2m^2\alpha - \frac{1}{F_1}\left[\frac{\sigma_1}{\epsilon}\left(-S_1\frac{\partial\beta}{\partial\xi} + S_2\frac{\partial\alpha}{\partial\xi}\right.\right.$$
$$\left.\left. -2\beta F_1\right) + m^2Q_2\left(S_1\frac{\partial\alpha}{\partial\xi} + S_2\frac{\partial\beta}{\partial\xi}\right)\right] = 0 \tag{55}$$

Equations (54) and (55) are linear. Their the solutions can take the form

$$\alpha(\xi, \tau) = Ae^{iq\xi+\delta\tau} + \text{C.C.} \tag{56}$$

and

$$\beta(\xi, \tau) = Be^{iq\xi+\delta\tau} + \text{C.C.} \tag{57}$$

Substituting (56) and (57) into (54) and (55), we get

$$\delta^2 + b_0\delta + (b_1 + ib_2) = 0 \tag{58}$$

where

$$b_0 = -2(2Q_2m^2 - q^2P_2) \tag{59}$$

$$b_1 = q^4(P_1^2 + P_2^2) + q^2[-2m^2(Q_1P_1 + 2Q_2P_2) - 2P_1\sigma_1\epsilon^{-1} + S_1^2 + S_2^2]$$
$$+ 4m^2(Q_1m^2\sigma_1\epsilon^{-1} + F_1^2 - \sigma_1^2\epsilon^{-2}) \tag{60}$$

$$b_2 = (2q/F_1)(\sigma_1\epsilon^{-1}S_1 - m^2Q_2S_2)(Q_1m^2 - \sigma_1\epsilon^{-1}) \tag{61}$$

The necessary and sufficient condition for stability requires that (Zahreddine and El Shehawey, 1988)

$$b_0 > 0 \tag{62}$$

$$(b_0^2 b_1 - b_2) > 0 \tag{63}$$

Conditions (62) and (63) lead to

$$\{2[F_1^2 - (\sigma_1^2/\epsilon^2)]^{1/2} - q^2 P_2\} > 0 \tag{64}$$

$$q^8 + a_1 q^6 + a_2 q^4 + a_3 q^2 + a_4 > 0 \tag{65}$$

where a_1, a_2, a_3, and a_4 are given in the Appendix.

The transition curves separating stable regions from unstable regions correspond to

$$q^2 = (2/P_2)[F_1^2 - (\sigma_1^2/\epsilon^2)]^{1/2} \tag{66}$$

$$q^8 + a_1 q^6 + a_2 q^4 + a_3 q^2 + a_4 = 0 \tag{67}$$

The transition curves (66) and (67) are plotted in the q^2–k plane for a sample case. The curves characterized by the symbols × and ○ represent the above transition curves, respectively.

Figure 1 represents a system with $\rho^{(1)} = 0.99823$ g/cm^3, $\rho^{(2)} = 0.7142$ g/cm^3, $\tilde{\epsilon}^{(1)} = 80$, $\tilde{\epsilon}^{(2)} = 55$, $T = 56$ dynes/cm, $g = 980.665$ cm/sec^2, $\omega_0 = 63$ Hz, $\epsilon g_0 = 9$, $E_0 = 0.0$, and $\alpha = 0.3$ g/cm^3 sec. The dashed line divides the graph into two regions, one corresponding to $\omega_0 < 2\omega_r$ and the other to $\omega_0 > 2\omega_r$, respectively. The solid vertical line divides the graph into regions corresponding to $F_1^2 - (\sigma_1^2/\epsilon^2) > 0$ and $F_1^2 - (\sigma_1^2/\epsilon^2) < 0$, respectively. The

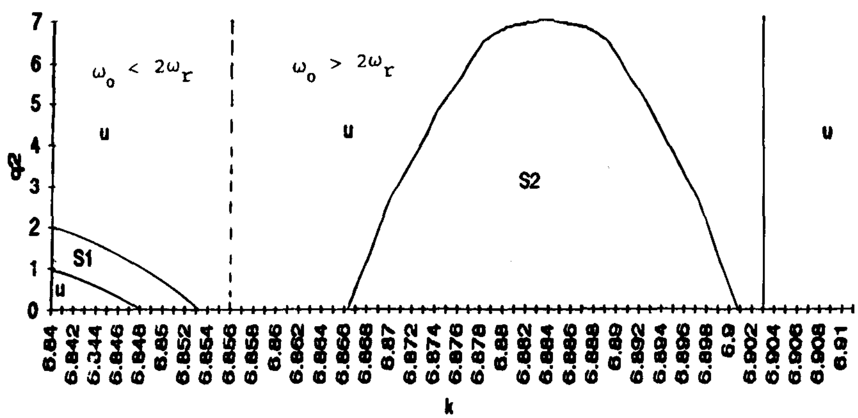

Fig. 1. A system with $\rho^{(1)} = 0.99823$ g/cm^3, $\rho^{(2)} = 0.7142$ g/cm^3, $\tilde{\epsilon}^{(1)} = 80$, $\tilde{\epsilon}^{(2)} = 55$, $T = 56$ dynes/cm, $g = 980$ cm/sec^2, $\omega_0 = 63$ Hz, $\epsilon g_0 = 9$, $E_0 = 0.0$, and $\alpha = 0.3$ g/cm^3 sec.

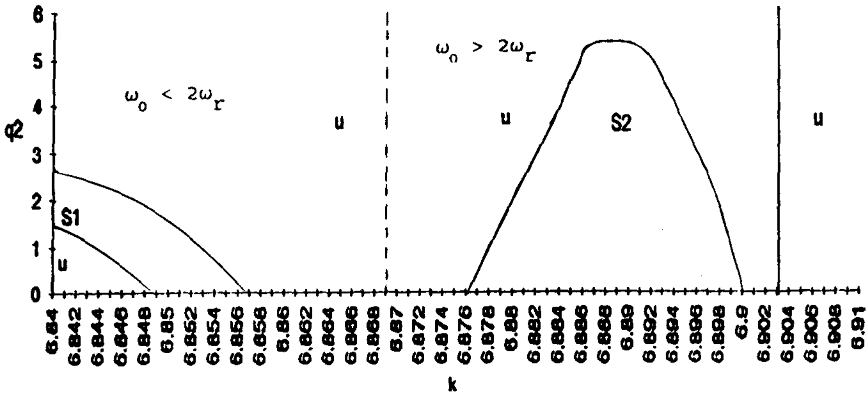

Fig. 2. The same system considered in Fig. 1, but with $\omega_0 = 64$ Hz.

region for which $F_1^2 - (\sigma_1^2/\epsilon^2) < 0$ is an unstable region, but the region for which $F_1^2 - (\sigma_1^2/\epsilon^2) > 0$ is stable if it satisfies the inequalities (64) and (65). The symbol u represents the unstable region. The regions S_1 and S_2 represent the stable regions for $\omega_0 < 2\omega_r$ and $\omega_0 > 2\omega_r$, respectively.

Figure 2 represents the same system as in Fig. 1, but with $\omega_0 = 64$ Hz. It is shown that the resonance point is shifted to the right-hand side. As ω_0 increases, the stable region S_1 increases, while S_2 decreases. It is clear that the frequency ω_0 plays a dual role in the stability criterion.

Figure 3 represents the same system as in Fig. 1, but with $E_0 = 0.4$ dynes/esu. It is clear that in the presence of an electric field the resonance point is shifted to the left. Also, the solid vertical line is shifted to the left. A comparison between Fig. 1 and 3 shows that, in the presence of an electric

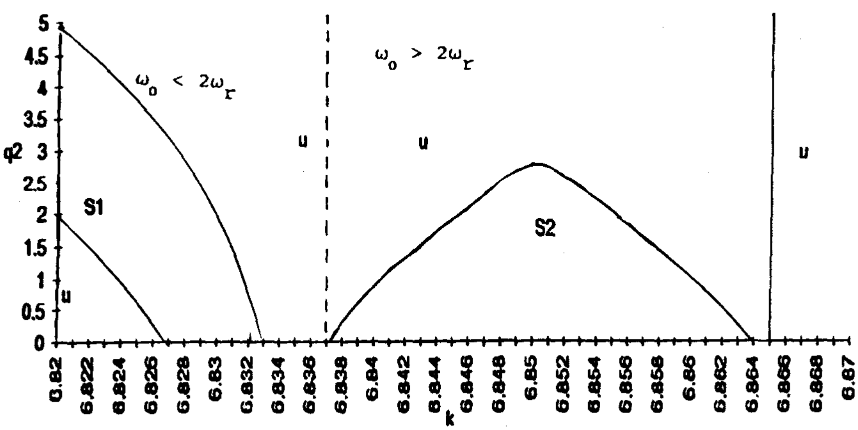

Fig. 3. The same system considered in Fig. 1, but with $E_0 = 0.4$ dynes/esu.

field, the stable region S_1 increases, while S_2 decreases. Thus the electric field plays a dual role in the stability criterion.

6. CONCLUSION

The nonlinear electrohydrodynamic Rayleigh–Taylor instability with mass and heat transfer is studied by imposing a vertical oscillating force and a horizontal electric field. The necessary and sufficient conditions for stability are obtained. Numerical calculations show that the external frequency and the electric field play dual role in the stability criterion.

APPENDIX

The following quantities are used in the text:

$$R = [-g_0^2k^2(\rho^{(1)} - \rho^{(2)})^2/4L_1^2\omega_r^2]\{[\omega_r/(\omega_0^2 - 4\omega_r^2)] + [(\omega_0 - \omega_r)/2(\omega_0 - 2\omega_r)] + (1/8\omega_r)\}$$

$$S_1 = [g_0k^2(\rho^{(1)} - \rho^{(2)})/4L_1^2\omega_r^2]([\omega_r/\omega_0(\omega_0 - 2\omega_r)] \times \{(-2M_3L_1/k)(\omega_0 - \omega_r) - (2\omega_0/k^2)(\omega_0 - \omega_r)[k(\rho^{(2)}h_2 \coth^2 kh_2 + \rho^{(1)}h_1 \coth^2 kh_1) - k(\rho^{(2)}h_2 + \rho^{(1)}h_1)] - [(\omega_0 - \omega_r)/k^2] \times [\alpha k(h_2 \coth^2 kh_2 + h_1 \coth^2 kh_1) - \alpha k(h_1 + h_2)] + (2\omega_r/k)(2\omega_1 L_1 - L_2)\} + (1/2)[(1/k^2)(L_2 - 2\omega_i L_1) + (h_1/k \sinh^2 kh_1) \times (-2\rho^{(1)}\omega_i + \alpha) + (h_2/k \sinh^2 kh_2)(-2\rho^{(2)}\omega_i + \alpha)])$$

$$S_2 = [g_0k^2(\rho^{(1)} - \rho^{(2)})/4L_1^2\omega_r^2]([\omega_r/\omega_0(\omega_0 - 2\omega_r)]\{(-2M_2L_1/k)(\omega_0 - \omega_r) - [(\omega_0 - \omega_r)^2 + \omega_i](1/k^2)[L_1 + k(\rho^{(2)}h_2 \coth^2 kh_2 + \rho^{(1)}h_1 \coth^2 kh_1) - k(\rho^{(2)}h_2 + \rho^{(1)}h_1)] - (\omega_i/k^2)[\alpha k(h_2 \coth^2 kh_2 + h_1 \coth^2 kh_1) - \alpha k(h_1 + h_2)] + [2(L_1\omega_r^2 + L_2\omega_i)/k^2] - (2g/k)(\rho^{(1)} - \rho^{(2)}) - [E_0^2/\epsilon^*(k)](\epsilon^{(2)} - \tilde{\epsilon}^{(1)})^2 \cosh kh_1 \cosh kh_2 - [E_0^2/\epsilon^{*2}(k)]k(\tilde{\epsilon}^{(2)}) - \tilde{\epsilon}^{(1)})^2(h_1\tilde{\epsilon}^{(1)} \times \cosh^2 kh_2 + h_2\tilde{\epsilon}^{(2)} \cosh^2 kh_1)\} - L_1D_1 + (1/2)[(2L_1\omega_r/k^2) + (2\rho^{(1)}\omega_r h_1 shk \sinh^2 kh_1) + (2\rho^{(2)}\omega_r h_2/k \sinh^2 kh_2)])$$

$$F_1 = [g_0k(\rho^{(1)} - \rho^{(2)})(\omega_0 - 2\omega_r)/8\epsilon\omega_r^2L_1]$$

$$M_2 = (1/2L_1\omega_r)\{(\omega_r^2 - \omega_i^2)[(\rho^{(1)}h_1/\sinh^2 kh_1) + (\rho^{(2)}h_2/\sinh^2 kh_2)$$
$$+ (L_1/k)] - \omega_i[(\alpha h_2/\sinh kh_2) + (\alpha h_1/\sinh kh_1) + (L_2/k)] + 2k\sigma$$
$$+ [E_0^2k(\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)})^2/\epsilon^{*2}(k)][-k(h_1\tilde{\epsilon}^{(1)} \cosh^2 kh_2 + h_2\tilde{\epsilon}^{(2)} \cosh^2 kh_1)$$
$$+ \cosh kh_1 \cosh kh_2]\}$$

$$M_3 = (1/2L_1)[(h_1/\sinh^2 kh_1)(2\omega_r\rho^{(1)} + \alpha) + (h_2/\sinh^2 kh_2)(2\omega_r\rho^{(2)} + \alpha)]$$

$$L_1 = (\rho^{(1)} \coth kh_1 + \rho^{(2)} \coth kh_2)$$

$$L_1 = \alpha(\coth kh_1 + \coth kh_2)$$

$$a_1 = [1/P_2^2(P_1^2 + P_2^2)]\{-4m^2Q_2P_2(P_1^2 + P_2^2) + P_2^2[-2m^2(Q_1P_1 + 2Q_2P_2)$$
$$- 2P_1\sigma_1\epsilon^{-1} + (S_1^2 + S_2^2)]\}$$

$$a_2 = [1/P_2^2(P_1^2 + P_2^2)]\{4m^2Q_2^2(P_1^2 + P_2^2)$$
$$- P_2^2[-2m^2(Q_1P_1 + 2Q_2P_2) - 4m^2Q_2P_2$$
$$\times [-2m^2(Q_1P_1 + 2Q_2P_2) - 2P_1\sigma_1\epsilon^{-1} + (S_1^2 + S_2^2)]$$
$$+ 4P_2^2(Q_1m^2\sigma_1\epsilon^{-1} + F_1^2 - \sigma_1^2\epsilon^{-2})\}$$

$$a_3 = [1/P_2^2(P_1^2 + P_2^2)]\{(4m^2Q_2^2 + P_2^2)[-2m^2(Q_1P_1 + 2Q_2P_2) - 2P_1\sigma_1\epsilon^{-1}$$
$$+ (S_1^2 + S_2^2)] - (1/F_1^2)(\sigma_1\epsilon^{-1}S_1 - m^2Q_2S_2)^2(m^2Q_1 - \sigma_1\epsilon^{-1})^2\}$$

$$a_4 = [1/P_2^2(P_1^2 + P_2^2)][16m^2Q_2(m^2Q_2 - P_2)(m^2Q_1\sigma_1\epsilon^{-1} + F_1^2 - \sigma_1^2\epsilon^2)]$$

and

$$\Theta = \Omega\{-(2i\alpha\alpha_2\omega/k)(\coth kh_2 + \coth kh_1)$$
$$+ i\alpha\omega(\coth^2 kh_2 + \coth^2 kh_1)$$
$$+ (\alpha^2/2\rho^{(1)}\sinh^2 kh_1) - (\alpha^2/2\rho^{(2)}\sinh^2 kh_2)$$
$$- \rho^{(2)}\omega^2[2 + (3/\sinh^2 kh_2)]$$
$$+ \rho^{(1)}\omega^2[2 + (3/\sinh^2 kh_1)] - [k^2E_0^2(\tilde{\epsilon}^2 - \tilde{\epsilon}^{(1)})/\epsilon^{*2}(k)\epsilon^*(2k)]$$
$$\times [-2\tilde{\epsilon}^{(1)}\tilde{\epsilon}^{(2)}\epsilon^*(k) \sinh 2k(h_1 + h_2) \sinh k(h_1 + h_2)$$
$$+ \epsilon^*(2k)\tilde{\epsilon}^{(1)}\tilde{\epsilon}^{(2)} \times \sinh^2 k(h_1 + h_2) - 4\epsilon^*(k)$$
$$\times (\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)})^2 \sinh 2kh_1 \sinh 2kh_2$$
$$\times \sinh kh_1 \sinh kh_2 - \epsilon^*(2k)(\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)})^2 \sinh^2 kh_1 \sinh^2 kh_2$$

$$+ 2\epsilon^*(k)(\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)})(\tilde{\epsilon}^{(2)}\cosh kh_2 \sinh 2kh_1 \sinh kh_1 \cosh 2kh_2$$
$$-\tilde{\epsilon}^{(1)}\sinh kh_2 \sinh 2kh_2 \cosh kh_1 \cosh 2kh_1) + \epsilon^{*^2}(k)\epsilon^*(2k)]\}$$
$$+ \{-3i\alpha\alpha_3(\coth kh_1 + \coth kh_2) + ik\alpha\omega[(3 - \coth^2 kh_2)\coth kh_2$$
$$+ (3 - \coth^2 kh_2)\coth kh_2] + \rho^{(2)}k\omega^2[2 - (2/\sinh^2 kh_2)]\coth kh_2$$
$$+ \rho^{(1)}k\omega^2[2 - (2/\sinh^2 kh_1)]\coth kh_1$$
$$- i\alpha\alpha_2\omega[(\cosh 2kh_2/\sinh^2 kh_2) - (\cosh 2kh_1/\sinh^2 kh_1)]$$
$$- \alpha^2 k[(\coth kh_1/\rho^{(1)}sinh^2 kh_1) + (\coth kh_2/\rho^{(2)}\sinh^2 kh_2)]$$
$$- (\alpha^2\alpha_2/\rho^{(1)})[3 - (\cosh 2kh_1/2 \sinh^2 kh_1)$$
$$- (\alpha^2\alpha_2/\rho^{(2)})[(\cosh 2kh_2/2 \sinh^2 kh_2) - 3] - (3/2)k^4\sigma$$
$$+ [k^3E_0^2(\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)})^2/\epsilon^{*^2}(k)\epsilon^*(2k)][-4\tilde{\epsilon}^{(1)}\tilde{\epsilon}^{(2)} \cosh 2kh_1$$
$$\times \cosh 2kh_2 \sinh^2 k(h_1 + h_2) + 6\tilde{\epsilon}^{(1)}\tilde{\epsilon}^{(2)} \sinh kh_1 \sinh kh_2$$
$$\times \sinh k(h_1 + h_2) \sinh 2k(h_1 + h_2)$$
$$+ 2(\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(2)}) \sinh kh_1 \sinh kh_2$$
$$\times (-\tilde{\epsilon}^{(2)} \cosh kh_2 \cosh 2kh_2 \sinh kh_1 \sinh 2kh_1 + \tilde{\epsilon}^{(1)} \cosh kh_1$$
$$\times \cosh 2kh_1 \sinh kh_2 \sinh 2kh_2) + 4(\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)})^2\sinh 2kh_2$$
$$\times \sinh 2kh_1 \sinh^2 kh_1 \sinh^2 kh_2 - 4\epsilon^*(k)\epsilon^*(2k) \sinh kh_1 \sinh kh_2]$$
$$+ 4i\alpha\alpha_2^2\omega[(1/\rho^{(2)}) \coth kh_2 - (1/\rho^{(1)}) \coth kh_1] + 2i\alpha_2\alpha\omega(\coth^2 kh_2$$
$$- \coth^2 kh_1) + 2\alpha_2\omega^2(\rho^{(2)}\coth^2 kh_2 - \rho^{(1)}\coth^2 kh_1) - [2E_0^2\alpha_2/\epsilon^{*^2}(k)]$$
$$\times (\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)})[k^2\tilde{\epsilon}^{(1)}\tilde{\epsilon}^{(2)} \sinh^2 k(h_1 + h_2) + (\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)})^2\sinh^2 kh_1$$
$$\times \sinh^2 kh_2 + 2\alpha^2\alpha_2[(1/\rho^{(1)}) - (1/\rho^{(2)})] - 2\alpha_2\omega^2(\rho^{(2)} - \rho^{(1)})$$
$$+ 2k^2\alpha_2 \times 2\, k^2\alpha_2 E_0^2\, (\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)})\}\, A^2\overline{A}$$
$$\Omega = \{-(i\alpha\alpha_2\omega/k)(\coth 2kh_2 + \coth 2kh_1) - 2i\omega\alpha(\coth 2kh_2 \coth kh_2$$
$$- \coth 2kh_1 \coth kh_1) + \omega^2(\rho^{(1)} \coth^2 kh_1 - \rho^{(2)} \coth^2 kh_2)$$
$$+ [2k^2E_0^2 \times (\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)})/\epsilon^{*^2}(k)\, \epsilon^*(2k)][\epsilon^*(k)\tilde{\epsilon}^{(1)}\tilde{\epsilon}^{(2)}\sinh k(h_1 + h_2)$$
$$\times \sinh 2k(h_1 + h_2) + (\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)})^2\epsilon^*(k) \sinh 2kh_1 \sinh 2kh_2$$
$$\times \sinh kh_1 \sinh kh_2 + (1/4)\epsilon^*(2k)(\tilde{\epsilon}^{(2)} - \tilde{\epsilon}^{(1)})(\tilde{\epsilon}^{(2)} \cosh 2kh_2$$

$$\times \sinh^2 kh_1 - \bar{\epsilon}^{(1)} \cosh 2kh_1 \sinh^2 kh_2) - \epsilon^{*2}(k)\epsilon^*(2k)]$$
$$+ (\alpha^2/\rho^{(1)}\rho^{(2)})(\rho^{(1)} - \rho^{(2)}) + (\alpha^2/2\rho^{(1)}\rho^{(2)})[(\rho^{(1)}/\sinh^2 kh_2)$$
$$- (\rho^{(2)}/\sinh^2 kh_1)] + (i\alpha\omega/\sinh^2 kh_2 \sinh^2 kh_1)(\sinh^2 kh_1$$
$$- \sinh^2 kh_2) + (\omega^2/2 \sinh^2 kh_2 \sinh^2 kh_1)[\rho^{(1)} \sinh^2 kh_2 - \rho^{(2)}$$
$$\times \sinh^2 kh_1]\}/\{(2\omega^2/k)(\rho^{(2)} \coth 2kh_2 + \rho^{(1)} \coth 2kh_1)$$
$$+ (i\alpha\omega/k)(\coth 2kh_2 + \coth 2kh_1) - [2kE_0/\epsilon^*(2k)]$$
$$\times (\bar{\epsilon}^{(2)} - \bar{\epsilon}^{(1)})^2 \sinh 2kh_1 \sinh 2kh_2 + g(\rho^{(2)} - \rho^{(1)}) - 4k^2\sigma\}$$

$$\epsilon^*(2k) = \bar{\epsilon}^{(1)} \cosh 2kh_2 \sinh 2kh_1 + \bar{\epsilon}^{(2)} \sinh 2kh_2 \cosh 2kh_1$$

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